Int. J. Solids Structures, 1974, Vol. 10 pp. 419–451. Pergamon Press. Printed in Gt. Britain.

BUCKLING AND DYNAMICS OF MULTILAYERED AND LAMINATED PLATES UNDER INITIAL STRESS†

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(*Received* 1 *September* 1973)

Abstract—The continuum mechanics of multilayered plates under initial stress is developed to include the case where some or all of the layers are constituted by thinly laminated materials with couple stresses. It is applicable to problems of buckling, dynamics and vibrations. This includes the evolution of viscoelastic creep buckling and vibration absorption. Results are obtained in two forms. One is derived from a rigorous analysis using the general theory of incremental deformations, the other provides drastic simplifications while retaining essential physical features which are consequences of the continuum behavior. Use of a particular definition of incremental stress is emphasized which is of special advantage in the type of problems under consideration. Corresponding variational principles are also formulated. Exact and approximate theories are compared both analytically and numerically. Excellent agreement is obtained.

1. INTRODUCTION

Based on the general theory of incremental deformations an exact solution was developed for multilayered structures under initial stress[1, 8]. The particular case of buckling for incompressible materials was discussed separately[3]. The layers were assumed orthotropic, elastic, or viscoelastic. The case of individual layers constituted by laminated composites was included using the elastic coefficients of the equivalent continuum. Extension of the theory taking into account couple stresses in laminated materials was developed in the context of static buckling and incompressible materials[4]. This extension also includes an added refinement referred to as "interstitial flow". The foregoing exact treatment is applicable to complex multilayered structures which may be embedded in an infinite continuum or may be free on one or both sides. In the latter case we are dealing with a multilayered plate. In this case it was shown[7, 9] that for a large variety of technological problems drastic simplifications may be introduced, in the analysis, without sacrificing essential physical features such as the skin effect or details of stress distribution required in the correct evaluation of damping, and of local damage due to stress concentrations. The purpose of the present analysis is two-fold. First to extend the exact dynamical theory under initial stress for compressible materials[1, 8], to include laminated layers with couple-stresses. Second to extend the approximate theory of multilayered plates[7, 9] with laminated layers and couple stresses to include a state of initial stress.

The fundamentals are presented in the context of plane strain deformation in a plane normal to the faces, since such solutions are immediately applicable to a large variety of

† This research has been sponsored by the A.F. Office of Scientific Research (N.A.M.), 1400 Wilson Boulevard, Arlington, Virginia 22209, through the European Office of Aerospace Research AFSC, United States Air Force, under contract F 44620-71-C-0092.

three-dimensional problems. The analysis is carried out using a definition of incremental stresses particularly suitable for the type of problem considered here. It was already used earlier occasionally[1, 4] and discussed more systematically in recent work[5]. This choice is a consequence of the fact that incremental stresses may be defined in various ways depending on the reference area before or after deformation and on the choice of the rotation of the local reference axes. The stress–strain relations and equilibrium equations as well as various definitions of incremental stresses are discussed in Sections 2 and 3, and in Appendix 1. The equilibrium equations are also derived by a suitably developed variational principle. Section 4 introduces couple-stresses in the formulation for the case of laminated layers, and a related variational principle. Section 5 shows the equations to be applicable to triaxial initial stresses. By using a couple-stress analogy derived earlier[4] the available exact solutions for homogeneous layers are immediately extended to include laminated materials with couple-stresses following the procedure outlined in Section 6 and Appendix 2. In Sections 7 and 8 drastic simplifications are obtained by extending to the case of initial stresses the approximate methods developed recently[7, 9]. It is shown how to formulate in simple form the characteristic equations for buckling and free oscillations. Resonance damping under initial stress is also evaluated. An important feature of these results is the practical validity of the expressions already derived earlier for resonance damping in the absence of initial stress. For transverse isotropic symmetry of the initial stresses and elastic properties the plane strain solutions are immediately applicable to the three-dimensional problems, for rectangular, triangular and circular plates. The procedure is briefly outlined in Section 9 following ideas developed earlier[9] in a more restricted context.

Exact elasticity solutions are compared with approximate results for the limiting case of large wavelengths in Section 10. For this case the exact solution becomes identical to the result obtained from the classical Euler-theory for the buckling of thin plates. A numerical comparison of the exact elasticity theory and the approximate analysis of Sections 7 and 8 is provided in Section 11. Excellent agreement is obtained. A comparison is also made with results derived from a "Timoshenko beam" approach.

All the results derived herein are applicable to both elastic and viscoelastic materials. Creep buckling analysis is readily obtained by methods discussed in detail earlier[1] for the case of viscoelasticity.

Approximate treatments of the mechanics of simple- and multilayered plates with initial stress have also been the object of attention by other investigators. Referring to the more recent contributions, the following may be cited. Srinivas and Kao[11] used approximate equations by adding membrane stresses to the classical theory of elasticity of an initially stress-free material. Viscous buckling of plates under various conditions was derived by De Leeuw[12, 13] and Mase[12]. Elastic buckling of anisotropic composite plates was studied by Chamis[14], Ashton[15] and Kicher and Mandel[16]. A survey of earlier studies of buckling of sandwich structures is given by Plantema[17].

2. BASIC EQUATIONS FOR INCREMENTAL DEFORMATIONS

We consider an orthotropic elastic plate whose faces are parallel to the x, z axes, with directions of elastic symmetry along x, y, z. The plate may be inhomogeneous, with or without discontinuities across the thickness, hence the elastic properties may be functions of y. For the sake of simplicity and intuitive clarity we start with the case of a biaxial initial stress represented by two principal components oriented along x and z. These components are

$$S_{11} = -P(y) \qquad S_{33} = S_{33}(y). \tag{2.1}$$

They are functions of y. The component ($S_{22} = 0$) normal to the faces is zero. The quantity $P(y)$ represents a compressive stress parallel to the faces and variable across the thickness. We shall see below that the analysis is easily extended to the case of triaxial initial stress where S_{22} is not zero.

It is important to note that the plate is assumed to be flat while already stressed initially. Deformations and stresses are considered as incremental quantities[1]. For incremental plane strain with displacements u, v in the x, y, plane, the first order strain components are

$$e_{xx} = \frac{\partial u}{\partial x} \qquad e_{yy} = \frac{\partial v}{\partial y} \qquad e_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right). \tag{2.2}$$

They generate incremental stresses

$$\begin{aligned} t_{11} &= C_{11}e_{xx} + C_{12}e_{yy} \\ t_{22} &= C_{12}e_{xx} + C_{22}e_{yy} \\ t'_{12} &= 2Le_{xy}. \end{aligned} \tag{2.3}$$

These stress–strain relations introduce four incremental elastic coefficients C_{11} C_{22} C_{12} and L which may be functions of y.

The incremental stress components t_{11} t_{22} t'_{12} were already defined and used earlier[1–4]. It is important however to recall their significance. This may be done in the following way. Consider a cube of material of unit size with edges oriented along x, y, z (Fig. 1a). We shall call it the unit element. This element is initially stressed by a compression $P = -S_{11}$ parallel to the x axis. There is also a principal initial stress S_{33} on the face normal to the z axes, but for plane strain in the x, y plane it does not appear explicitly in the formulation. Note that the four elastic coefficients in the stress–strain relations (2.3) will generally depend not only on the physical nature of the material but also on the state of initial stress hence on the two

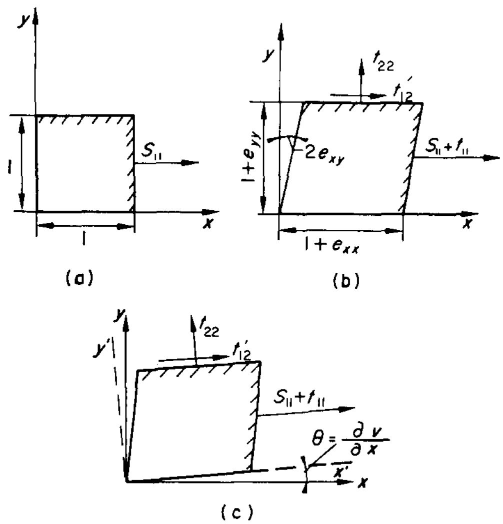

Fig. 1. Definition of incremental stresses.

stress components, $S_{11}(y)$ and $S_{33}(y)$. We now apply stress increments to the unit element. Consider first a plane strain deformation without shear ($e_{xy} = 0$). This requires the application of principal stresses $S_{11} + t_{11}$ in the x direction and t_{22} in the y direction (Fig. 1b). These stresses are actually forces applied to the unit element so that t_{11} and t_{22} define incremental *stresses per unit initial area.* The incremental stresses are assumed to be small of the first order so that t_{22} may be referred indifferently to initial or final area. The principal strains e_{xx} and e_{yy} generated by the stress increments are also small, and they are related linearly to the stress-components t_{11} and t_{22} by the first two of equations (2.3). Next we apply a tangential stress t'_{12} in the x direction on the face normal to the y axis, in such a way that this face does not rotate but slides with respect to the other parallel face which remains along the x axis. (Fig. 1b.) The deformation is represented by a shear strain $2e_{xy}$, related to t'_{12} by the last of equations (2.3) with the elastic coefficient L. This coefficient L was called the *slide modulus*[1–4] to distinguish it from the classical shear modulus.

Obviously after these incremental deformations of the unit element have occurred we may rotate it rigidly through a small angle

$$\theta = \frac{\partial v}{\partial x} \tag{2.4}$$

about an axis normal to the x, y plane. Since there are no additional deformations occurring during this rotation the whole stress system including both initial and incremental stresses rotate with it. Hence S_{11}, t_{11}, t_{22}, t'_{12} are now referred to axes $x'y'$ obtained by rotating the original axes x, y through an angle θ (Fig. 1c). An important property enters into play here regarding strain components. Since they are assumed small as well as the rotation their value referred to the rotated axes are the same as those given by equations (2.2) where u, and v are the projections of the displacements on the fixed initial axes. Hence the deformed and rotated unit element as described above may be considered to belong to a continuous deformation field where $e_{xx}\,e_{yy}\,e_{xy}$ and $t_{11}t_{22}\,t'_{12}$ are the local strains and incremental stresses referred to axes rotated locally through an angle $\theta = \partial v/\partial x$.

The equilibrium equations for the stress-field are written

$$\begin{aligned} \frac{\partial t_{11}}{\partial x} + \frac{\partial t'_{12}}{\partial y} &= \rho\,\frac{\partial^2 u}{\partial t^2} \\ \frac{\partial t'_{12}}{\partial x} + \frac{\partial t_{22}}{\partial y} &= \rho\,\frac{\partial^2 v}{\partial t^2} + P\,\frac{\partial^2 v}{\partial x^2} \end{aligned} \tag{2.5}$$

where $\rho(y)$ is the specific mass which may be a function of y. These equilibrium equations with the addition of inertia terms are the same as obtained in earlier work[1–4]. They have the advantage of an obvious intuitive interpretation. Several alternate derivations of these equations are also given in Appendix 1, including the more general case with four components S_{11} S_{22} S_{12} S_{33} for the initial stresses.

3. ON VARIOUS REPRESENTATIONS OF THE INCREMENTAL STRESS

Before proceeding any further it is useful to point out that incremental stresses may be defined in a variety of ways. The choice is usually a matter of convenience in the particular problem considered. First there is the matter of reference area for the stresses. They may be referenced to initial areas before deformation. Second the choice of the local rotation for the axes x', y' to which local stresses and strains are referred is not unique. This was already pointed out earlier[1] and discussed in more detail in a recent paper[5]. Let us discuss these various representations in the present, more restricted plane strain context, where the initial stress is restricted to the two principal stresses $S_{11} = -P$ and S_{33} while $S_{22} = 0$. Consider again the local rotation to be defined by (2.4), i.e. $\theta = \partial v/\partial x$. The stress components in the locally rotated axes $x'y'$ may be referred to areas after deformation. We shall denote these components by s'_{11} s'_{22} s'_{12}. Obviously since we are dealing with first order quantities we may write

$$\begin{aligned} t_{11} &= s'_{11} + S_{11}e_{yy} = s'_{11} - Pe_{yy} \\ t_{22} &= s'_{22} \\ t'_{12} &= s'_{12}. \end{aligned} \tag{3.1}$$

On the other hand a different angle of local rotation may be used and defined as

$$\omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right). \tag{3.2}$$

The stresses referred to final areas and to locally rotated axes through the angle ω have been denoted by s_{11} s_{22} s_{12}[1, 6]. The relation between the stresses s'_{ij} and s_{ij} is to the first order

$$\begin{aligned} s_{11} &= s'_{11} \\ s_{22} &= s'_{22} \\ s_{12} &= s'_{12} - P\left(\frac{\partial v}{\partial x} - \omega\right) = s'_{12} - Pe_{xy}. \end{aligned} \tag{3.3}$$

The local rotation ω and the incremental stresses s_{ij} have been used extensively in the general theory of incremental deformation[1, 6]. They are also discussed in Appendix 1. They provide mathematical symmetry in the general equations. They are also more convenient to describe the physical properties of isotropic materials. However for the present case of orthotropic symmetry the stress system t_{11} t_{22} t'_{12} provides a simpler physical description and analysis as well as a convenient intuitive interpretation.

From equations (3.3) we may derive the values of s'_{11} s'_{22} s'_{12} and substitute these values in equations (3.1). We obtain

$$\begin{aligned} t_{11} &= s_{11} - Pe_{yy} \\ t_{22} &= s_{22} \\ t'_{12} &= s_{12} + Pe_{xy}. \end{aligned} \tag{3.4}$$

In earlier work[1, 6] we have also introduced stress–strain relations in terms of the stresses s_{ij}. They are

$$\begin{aligned} s_{11} &= B_{11}e_{xx} + B_{12}e_{yy} \\ s_{22} &= B_{21}e_{xx} + B_{22}e_{yy} \\ s_{12} &= 2Qe_{xy}. \end{aligned} \tag{3.5}$$

It was shown that the existence of an elastic potential implies the relation

$$B_{12} = B_{21} + P. \tag{3.6}$$

Hence in general $B_{12} \neq B_{21}$. By substituting these values (3.5) in equations (3.4) we obtain

$$\begin{aligned} t_{11} &= B_{11} e_{xx} + (B_{12} - P) e_{yy} \\ t_{22} &= B_{21} e_{xx} + B_{22} e_{yy} \\ t'_{12} &= (2Q + P) e_{xy}. \end{aligned} \tag{3.7}$$

Comparison with the stress–strain relations (2.3) yields the following relations between the two types of elastic coefficients

$$\begin{aligned} C_{11} &= B_{11} \qquad C_{22} = B_{22} \\ C_{12} &= B_{21} = B_{12} - P \\ L &= Q + \tfrac{1}{2}P. \end{aligned} \tag{3.8}$$

4. LAMINATED MEDIA, WITH INITIAL STRESS AND COUPLE STRESSES

Consider a laminated plate constituted by an alternation of thin hard and soft layers. The hard layer has a thickness h'_1 and the soft layer a thickness h'_2. The fractions of the total thickness occupied by each type of layer are

$$\alpha_1 = \frac{h'_1}{h'_1 + h'_2} \qquad \alpha_2 = \frac{h'_1}{h'_1 + h'_2}. \tag{4.1}$$

Under certain limitations already discussed earlier [1, 3, 4, 7], such a laminated medium may be replaced by a continuum with certain average elastic coefficients. In addition we must also introduce couple-stresses.

As in the foregoing analysis we assume that the initial stress on planes normal to the y axis is zero ($S_{22} = 0$). The x components of the initial stress in the hard and soft layer are denoted respectively by $S_{11}^{(1)} = -P_1$ and $S_{11}^{(2)} = -P_2$.

The stress–strain relations of the equivalent continuum retain the same form as in (2.3) where the strain components and incremental stresses are averaged values. The corresponding elastic coefficients for the equivalent continuum for a compressible medium were evaluated earlier[1]. The following results were obtained. First we express the incremental stress–strain relations for the hard layer. They are of the form

$$\begin{aligned} t_{11} &= \mathscr{A}_1 e_{xx} + \mathscr{B}_1 e_{yy} \\ t_{22} &= \mathscr{B}_1 e_{xx} + \mathscr{C}_1 e_{yy} \\ t'_{12} &= 2L_1 e_{xy} \end{aligned} \tag{4.2}$$

where $\mathscr{A}_1$ $\mathscr{B}_1$ $\mathscr{C}_1$ L_1 are the four elastic coefficients for the hard layer with the same meaning as in equations (2.3). For the soft layer we write similarly

$$\begin{aligned} t_{11} &= \mathscr{A}_2 e_{xx} + \mathscr{B}_2 e_{yy} \\ t_{22} &= \mathscr{B}_2 e_{xx} + \mathscr{C}_2 e_{yy} \\ t'_{22} &= 2L_2 e_{xy} \end{aligned} \tag{4.3}$$

with the four incremental coefficients $\mathscr{A}_2$ $\mathscr{B}_2$ $\mathscr{C}_2$ L_2 corresponding to that layer. The four average coefficients of the equivalent continuum as derived in [1] are

$$
\begin{aligned}
C_{11} &= \alpha_1 \mathscr{A}_1 + \alpha_2 \mathscr{A}_2 - \frac{\alpha_1 \alpha_2 (\mathscr{B}_1 - \mathscr{B}_2)^2}{\alpha_1 \mathscr{C}_1 + \alpha_2 \mathscr{C}_2} \\
C_{12} &= \frac{\alpha_1 \mathscr{B}_1 \mathscr{C}_2 + \alpha_2 \mathscr{B}_2 \mathscr{C}_1}{\alpha_1 \mathscr{C}_2 + \alpha_2 \mathscr{C}_1}. \\
C_{22} &= \frac{\mathscr{C}_1 \mathscr{C}_2}{\alpha_1 \mathscr{C}_2 + \alpha_2 \mathscr{C}_1} \\
L &= 1 \Big/ \left(\frac{\alpha_1}{L_1} + \frac{\alpha_2}{L_2} \right).
\end{aligned} \tag{4.4}
$$

These are the coefficients to be substituted in the stress–strain relations (2.3), in order to express the average properties of the equivalent continuum.

Note that in the equilibrium equations (2.5) u and v are also average displacements and

$$\rho = \alpha_1 \rho_1 + \alpha_2 \rho_2 \tag{4.5}$$

is the average specific mass, with ρ_1, ρ_2 representing the specific mass of each layer. Furthermore we must replace P by

$$P = \alpha_1 P_1 + \alpha_2 P_2 \tag{4.6}$$

in terms of the initial stress P_1 and P_2 in each layer.

If the hard layer is sufficiently stiff a couple-stress of moment $\mathscr{M}$ per unit area is produced in the plane normal to the x axis. As indicated earlier[4, 7] the stress t'_{12} in the second equilibrium equation (2.5) must then be replaced by $t'_{21} \neq t'_{12}$. Hence equations (2.5) become

$$
\begin{aligned}
\frac{\partial t_{11}}{\partial x} + \frac{\partial t'_{12}}{\partial y} &= \rho \frac{\partial^2 u}{\partial t^2} \\
\frac{\partial t'_{21}}{\partial x} + \frac{\partial t_{22}}{\partial y} &= \rho \frac{\partial^2 v}{\partial t^2} + P \frac{\partial^2 v}{\partial x^2}.
\end{aligned} \tag{4.7}
$$

Equilibrium of moments for an element of material requires the condition

$$t'_{12} - t'_{21} = \frac{\partial \mathscr{M}}{\partial x}. \tag{4.8}$$

The value of t'_{21} derived from this condition is then substituted in (4.7) thus yielding equilibrium equations in the form

$$
\begin{aligned}
\frac{\partial t_{11}}{\partial x} + \frac{\partial t'_{12}}{\partial y} &= \rho \frac{\partial^2 u}{\partial t^2} \\
\frac{\partial t'_{12}}{\partial x} + \frac{\partial t_{22}}{\partial y} &= \rho \frac{\partial^2 v}{\partial t^2} + P \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 \mathscr{M}}{\partial x^2}.
\end{aligned} \tag{4.9}
$$

The value of the moment $\mathscr{M}$ in terms of the deformation was obtained earlier[4, 7]. It was found that

$$\mathscr{M} = b \frac{\partial^2 v}{\partial x^2} \tag{4.10}$$

where the couple stress coefficient is†

† Note the misprint in [4].

$$b = \tfrac{1}{3}h'^2\alpha_1\alpha_2 \frac{(L_1 - L_2)}{\alpha_1 L_1 + \alpha_2 L_2}(M_1\alpha_1^2 - M_2\alpha_2^2) \tag{4.11}$$

with

$$\begin{aligned} h' &= h_1' + h_2' \\ M_1 &= \frac{1}{4\mathscr{C}_1}(\mathscr{A}_1\mathscr{C}_1 - \mathscr{B}_1^2) \\ M_2 &= \frac{1}{4\mathscr{C}_2}(\mathscr{A}_2\mathscr{C}_2 - \mathscr{B}_2^2). \end{aligned} \tag{4.12}$$

If one layer is much more rigid that the other the couple stress coefficient has the simple value

$$b = \tfrac{1}{3}h'^2 M \tag{4.13}$$

where $M = M_1\alpha_1$ (see [4]).

Substituting the value (4.10) for $\mathscr{M}$ the equilibrium equations (4.9) become

$$\begin{aligned} \frac{\partial t_{11}}{\partial x} + \frac{\partial t'_{12}}{\partial y} &= \rho \frac{\partial^2 u}{\partial t^2} \\ \frac{\partial t'_{12}}{\partial x} + \frac{\partial t_{22}}{\partial y} &= \rho \frac{\partial^2 v}{\partial t^2} + P\frac{\partial^2 v}{\partial x^2} + b\frac{\partial^4 v}{\partial x^4}. \end{aligned} \tag{4.14}$$

Variational principle

Equations (4.13) are obviously equivalent to the variational principle (for arbitrary variations δu and δv),

$$\delta \int \Delta V \, dx\, dy + \int \rho(\ddot{u}\,\delta u + \ddot{v}\,\delta v)\, dx\, dy = 0 \tag{4.15}$$

with $\ddot{u} = \partial^2 u/\partial t^2$, $\ddot{v} = \partial^2 v/\partial t^2$ and

$$\Delta V = \tfrac{1}{2}t_{11}e_{xx} + \tfrac{1}{2}t_{22}e_{yy} + t'_{12}e_{xy} - \tfrac{1}{2}P\left(\frac{\partial v}{\partial x}\right)^2 + \tfrac{1}{2}b\left(\frac{\partial^2 v}{\partial x^2}\right)^2. \tag{4.16}$$

Substituting the values (2.3) for the stress we obtain

$$\begin{aligned} \Delta V = {} & \tfrac{1}{2}C_{11}e_{xx}^2 + \tfrac{1}{2}C_{22}e_{yy}^2 + C_{12}e_{xx}e_{yy} \\ & + 2Le_{xy}^2 - \tfrac{1}{2}P\left(\frac{\partial v}{\partial x}\right)^2 + \tfrac{1}{2}b\left(\frac{\partial^2 v}{\partial x^2}\right)^2. \end{aligned} \tag{4.17}$$

The variational principle (4.16) is a generalization of the principle derived earlier[4] for the incompressible material. It also constitutes a particular case of the general variational principle of earlier theories[1, 6] and further discussed in the Appendix 1, to which we then add the couple-stress term. Note that by introducing the kinetic energy the variational principle (4.15) may be written

$$\delta \int dt \int [\rho(u^2 + \dot{v}^2) - \Delta V]\, dx\, dy = 0 \tag{4.18}$$

which is of Hamiltonian form.

Laminated plates with non-homogeneous average properties

It should be pointed out that the foregoing results are applicable to plates, with different types of laminations across the thickness. In this case the average moduli (4.4) and the couple-stress coefficient of equation (4.11) as well as P will be functions of y. In particular the plate may be composed of various homogeneous layers, while other layers are constituted by laminated materials each with its own average moduli and couple-stress coefficient.

5. EXTENSION TO TRIAXIAL INITIAL STRESS

Until now we have assumed that the initial stress is represented by the two principal components (2.1) while the component normal to the plate is zero ($S_{22} = 0$). However the solutions for this case are completely general and may be extended immediately without further calculation to the case of triaxial initial stress provided the variables are suitably interpreted physically[1]. Consider initial stresses represented by the three principal components

$$S_{11} = S_{11}(y) \qquad S_{22} = \text{const.} \qquad S_{33} = S_{33}(y). \tag{5.1}$$

The component S_{22} normal to the plate is assumed constant.

This state of triaxial stress may be generated by starting from the biaxial state of components (2.1) and immersing the whole system in a uniform hydrostatic field of constant pressure p_f. In this case the initial stresses become

$$\begin{aligned} S_{11} &= -P - p_f \\ S_{22} &= -p_f \\ S_{33} &= S'_{33} - p_f. \end{aligned} \tag{5.2}$$

The stress-strain relations (2.3) retain the same form. The incremental elastic coefficients may now depend on all three components S_{11}, S_{22}, and S_{33} of the initial stress. The equilibrium equations (2.5) are also unaffected by superposing a hydrostatic stress field. The value of P in these equations is now

$$P = S_{22} - S_{11} \tag{5.3}$$

which represents an effective compression. However we must be careful to interpret the stresses t_{11} and t_{22} correctly. They do not represent the actual stress increment per unit initial area, but only that portion of the stress increment which is not due to the hydrostatic stress. For example the actual stress increments along x and y are respectively

$$\begin{aligned} t'_{11} &= t_{11} - p_f e_{yy} = t_{11} + S_{22} e_{yy} \\ t'_{22} &= t_{22} - p_f e_{xx} = t_{22} + S_{22} e_{xx}. \end{aligned} \tag{5.4}$$

The components t_{11} t_{22} are the more significant ones physically since they represent the applied forces when the material is tested inside a chamber containing a fluid at the pressure p_f.

That the equilibrium equations (2.5) remain the same for triaxial initial stress was shown earlier[1]. It may also be verified by substituting the values (5.4) for t'_{11} and t'_{22} into the equilibrium equations (1.24) derived rigorously in Appendix 1 for initial stresses of a more general type. Putting $S_{12} = 0$ we obtain the equilibrium equations (2.5) with $P = S_{22} - S_{11}$.

Under the same conditions the equilibrium equations (4.14) for a material with couple stresses are also valid for the case of triaxial initial stress.

6. EXACT SOLUTIONS FOR MULTILAYERS. COUPLE-STRESS ANALOGY

We first consider a single homogeneous plate of thickness h along y, subject to a uniform initial compressive stress, $P = -S_{11}$ along the x direction. The normal initial stress is assumed to be zero ($S_{22} = 0$). However this assumption is not restrictive since we have seen in the preceding section how the results are readily applicable to the more general case of triaxial initial stress. Complete solutions for this case were derived[1, 8] for harmonic oscillations and a sinusoidal deformation along x. In this case the displacement field and the incremental stresses are of the form†

$$\begin{aligned} u &= U(ly)\sin lx \qquad & v &= V(ly)\cos lx \\ t'_{12} &= \tau(ly)\sin lx \qquad & t_{22} &= q'(ly)\cos lx . \end{aligned} \tag{6.1}$$

For harmonic oscillations these quantities should be multiplied by a time factor $\exp(i\alpha t)$ which is omitted here. The circular frequency is denoted by α. If $y = h/2$ and $y = -h/2$ represent respectively the top and bottom faces we put

$$\begin{aligned} U_1 &= U(lh/2) \quad & V_1 &= V(lh/2) \quad & U_2 &= U(-lh/2) \quad & V_2 &= V(-lh/2) \\ \tau_1 &= \tau(lh/2) \quad & q'_1 &= q'(lh/2) \quad & \tau_2 &= \tau(-lh/2) \quad & q'_2 &= q'(-lh/2). \end{aligned} \tag{6.2}$$

The four variables τ_1 q'_1 τ_2 q'_2 represent the stresses applied to the faces of the plate. They are linear functions of the displacements U_1 V_1 U_2 V_2 of these faces. This relationship was derived earlier[1, 8] in the following form

$$\begin{aligned} \tau_1 &= lL\frac{\partial I}{\partial U_1} \qquad & \tau_2 &= -lL\frac{\partial I}{\partial U_2} \\ q'_1 &= lL\frac{\partial I}{\partial V_1} \qquad & q'_2 &= -lL\frac{\partial I}{\partial V_2} \end{aligned} \tag{6.3}$$

where

$$\begin{aligned} I = {} & \tfrac{1}{2}A(U_1^2 + U_2^2) - DU_1U_2 \\ & + \tfrac{1}{2}C(V_1^2 + V_2^2) + FV_1V_2 \\ & + B(U_1V_1 - U_2V_2) + E(U_1V_2 - U_2V_1). \end{aligned} \tag{6.4}$$

These equations contain six basic coefficients A B C D E F which constitute the elements of a four by four matrix. These coefficients are functions of the four elastic moduli of the plate, C_{11} C_{22} C_{12} L, of the plate thickness h, the wave number l, the frequency α, the density ρ, and the initial stress P.

Couple-stress analogy

This result may be immediately extended, to a laminated plate with couple-stresses. First the elastic coefficients are put equal to expressions (4.4) which are those of the equivalent continuum. The density is also replaced by the average value (4.5). In addition the effect of couple-stresses may be introduced by the following very simple procedure. We note that for

† In references [1, 8] the notation q is used instead of q'.

a sinusoidal and harmonic field of the form (6.1) the equilibrium equations (4.14) may be written

$$\frac{\partial t_{11}}{\partial x} + \frac{\partial t'_{12}}{\partial y} + \rho\alpha^2 u = 0$$
$$\frac{\partial t'_{12}}{\partial x} + \frac{\partial t'_{22}}{\partial y} + \rho\alpha^2 v = (P - bl^2)\frac{\partial^2 v}{\partial x^2}. \tag{6.5}$$

Hence the effect of couple-stresses represented by the coefficient b amounts to a replacement of P by $P - bl^2$ in all solutions for which $b = 0$. This property referred to as the couple-stress analogy was derived earlier[4] for the particular case of static deformations and incompressible materials. It is hereby extended to the dynamics of compressible materials. Values of the coefficients $A\ B\ C\ D\ E\ F$ using the couple-stress analogy are derived in Appendix 2, for a plate with couple stresses using the expressions obtained earlier[1, 8] without couple stresses.

Equations for multilayers

We consider a plate constituted by an arbitrary number of layers some of which or all may be laminated hence producing couple-stresses. Each layer is assumed homogeneous with respect, to its average properties, i.e. within this layer, the average elastic coefficients of the equivalent continuum, the stress-couple coefficient b as well as the average density ρ and the initial stress P are all constant. We number these layers from 1 to n and consider the ith layer. The six matrix elements for these layers are denoted by $A_i\ B_i\ C_i\ D_i\ E_i\ F_i$. Complete interfacial adherence is assumed. We denote by $U_i\ V_i$ the displacement at the interface between layer i and $i - 1$ and define I_i as

$$\begin{aligned} I_i = {} & \tfrac{1}{2}A_i(U_i^2 + U_{i+1}^2) - D_i U_i U_{i+1} \\ & + \tfrac{1}{2}C_i(V_i^2 + V_{i+1}^2) + F_i V_i V_{i+1} \\ & + B_i(U_i V_i - U_{i+1}V_{i+1}) + E_i(U_i V_{i+1} - U_{i+1}V_i). \end{aligned} \tag{6.6}$$

It was shown[1, 8] that the condition of continuity of the stresses $\tau\ q'$ at the interface between layers i and $i + 1$ is

$$\frac{\partial}{\partial U_{i+1}}(L_i I_i + L_{i+1}I_{i+1}) = 0$$
$$\frac{\partial}{\partial V_{i+1}}(L_i I_i + L_{i+1}I_{i+1}) = 0. \tag{6.7}$$

On the other hand the stresses $\tau_1\ q'_1$ at the top surface and τ_{n+1}, q'_{n+1} at the bottom surface of the multilayer are

$$\tau_1 = lL_1\frac{\partial I_1}{\partial U_1} \qquad q'_1 = lL_1\frac{\partial I_1}{\partial V_1}$$
$$\tau_{n+1} = -lL_n\frac{\partial I_n}{\partial U_{n+1}} \qquad q'_{n+1} = -lL_n\frac{\partial I_n}{\partial V_{n+1}}. \tag{6.8}$$

For given applied stresses at top and bottom of the multilayered plate equations (6.7) and (6.8) constitute a system of $2n + 2$ equations for the $2n + 2$ displacements $U_i\ V_i$. Equations

(6.7) are two recurrence equations between six displacements at three consecutive interfaces. When the multilayer is free at the top and bottom we put $\tau_1 = q_1' = \tau_{n+1} = q_{n+1}' = 0$, and the system is homogeneous. The characteristic determinant provides the natural frequency α of oscillation under initial stress. The condition $\alpha = 0$ yields the initial stress P of buckling instability. For larger values of P the exponential coefficient $p = i\alpha$ becomes real and dynamic buckling appears. Note the variational principle implicit in equations (6.7) and (6.8).

7. SIMPLIFIED APPROXIMATE THEORY

A simplified approximate theory of multilayered plates was developed earlier[7, 9] and may be readily extended to plates under initial stresses. The procedure remains essentially the same. The purpose of this approximate theory is to provide a treatment which is drastically simpler than the exact elasticity theory outlined in the preceding section, while at the same time retaining essential physical features such as the skin effect[7] or certain details of internal stress distribution required for a realistic predictions in design problems. The accuracy of this approximate analysis for buckling problems is evaluated numerically in Section 11 below.

A basic simplifying assumption is

$$t_{22} = 0. \tag{7.1}$$

The stress–strain relations (2.3) become

$$\begin{aligned} t_{11} &= 4Me_{xx} \\ t'_{12} &= 2Le_{xy} \end{aligned} \tag{7.2}$$

where

$$M = \frac{1}{4C_{22}}(C_{11}C_{22} - C_{12}^2). \tag{7.3}$$

The coefficients M, L may be functions of the coordinate y across the thickness. The displacement field and shear stresses are put equal to

$$\begin{aligned} u &= U(y)\sin lx \\ v &= V\cos lx \\ t'_{12} &= \tau(y)\sin lx. \end{aligned} \tag{7.4}$$

For harmonic oscillations of circular frequency α the time factor $\exp(i\alpha t)$ has been omitted.

The second simplifying assumption is introduced in these expressions by treating V as a constant equal to the average value along y.

For a laminated plate constituted by an alternation of hard and soft thin layers the two elastic coefficients of these layers are denoted respectively by M_1 L_1 and M_2 L_2. The coefficients for the average equivalent continuum are

$$\begin{aligned} M &= M_1\alpha_1 + M_2\alpha_2 \\ L &= 1\Big/\left(\frac{\alpha_1}{L_1} + \frac{\alpha_2}{L_2}\right) \end{aligned} \tag{7.5}$$

where α_1 and α_2 are the fractions of total thickness occupied by each layer.

In terms of the coefficients $\mathscr{A}_1\mathscr{B}_1\mathscr{C}_1$ and $\mathscr{A}_2\mathscr{B}_2\mathscr{C}_2$ for each lamination the values of M_1 and M_2 are expressed by equations (4.12). It may be verified that by substituting the values (4.4) of $\mathscr{C}_{11}\mathscr{C}_{22}\mathscr{C}_{12}$ into expression (7.3) for M the result is identical to that obtained from equation (7.5).

A couple-stress coefficient b given by equation (4.11) is also introduced. The average initial stress P and density ρ of the laminated medium are expressed by (4.5) and (4.6). Note that average parameters $M\,L\,b\,P$ and ρ may be function of y. From equations (7.2) and (7.4) we derive

$$\tau(y) = L\left(\frac{\mathrm{d}U}{\mathrm{d}y} - lV\right). \tag{7.6}$$

Combining equations (7.2) (7.4) (7.6) and the first of the equilibrium equations (2.5) we derive

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{1}{4\mathfrak{M}}\frac{\mathrm{d}\tau}{\mathrm{d}y}\right) - \frac{l^2}{L}\tau = l^3 V \tag{7.7}$$

where

$$\mathfrak{M} = M\left(1 - \frac{\rho\alpha^2}{4Ml^2}\right). \tag{7.8}$$

We also derive

$$\begin{aligned} U &= \frac{1}{4\mathfrak{M}l^2}\frac{\mathrm{d}\tau}{\mathrm{d}y} \\ t_{11} &= \frac{M}{\mathfrak{M}l^2}\frac{\mathrm{d}\tau}{\mathrm{d}y}\cos lx. \end{aligned} \tag{7.9}$$

These equations are identical to those obtained for the approximate theory of the plate *without initial stress*[9]. The problem is reduced to solving the Sturm–Liouville equation (7.7). The solution is determined by the boundary conditions for τ and the top and bottom of the plate, and contains an unknown deflection V. The latter is obtained from the second equilibrium equation (2.5). By integrating this equation with respect to y we obtain

$$q = -\int_{-h/2}^{+h/2} \tau(y)\,\mathrm{d}y - (\alpha^2\rho_t + P_t l^2 - b_t l^4)V. \tag{7.10}$$

In this expression $y = \pm h/2$ represent the top and bottom of the plate, while

$$[t_{22}]_{y=h/2} - [t_{22}]_{y=-h/2} = q\cos lx \tag{7.11}$$

is the total normal load on the plate. Other quantities are

$$\begin{aligned} \rho_t &= \int_{-h/2}^{+h/2} \rho(y)\,\mathrm{d}y \\ P_t &= \int_{-h/2}^{+h/2} P(y)\,\mathrm{d}y \\ b_t &= \int_{-h/2}^{+h/2} b(y)\,\mathrm{d}y. \end{aligned} \tag{7.12}$$

Basic analogy

We express the value (7.10) of q by separating it into two parts

$$q \doteq \mathscr{S} - \mathscr{R}V \tag{7.13}$$

where

$$\mathscr{S} = -\int_{-h/2}^{+h/2} \tau(y)\,\mathrm{d}y$$

$$\mathscr{R} = \alpha^2 \rho_t + P_t l^2 - b_t l^4. \tag{7.14}$$

We note that for $\alpha = P_t = b_t = 0$,

$$q = \mathscr{S} \tag{7.15}$$

represents the load for the static solution in the absence of initial stress and couple-stress. Hence in order to obtain the dynamic solution with initial and couple stresses all we need to do is to replace M by $\mathfrak{M}$ in the static solution (7.15) and replace q by $q + \mathscr{R}V$. In the context of the approximate theory this property constitutes a *basic analogy* by which static solutions may be immediately extended to the more general case without further calculations. The couple stress analogy of section 6 is a particular case of this more general property of the simplified theory. In many cases a further simplification is obtained simply by putting

$$\mathfrak{M} = M \tag{7.16}$$

thus maintaining the static solution (7.15) without replacing M by $\mathfrak{M}$.

8. METHODS OF SIMPLIFIED ANALYSIS OF BUCKLING VIBRATIONS AND VISCOELASTIC DAMPING

As shown in the preceding section the approximate simplified analysis is readily applied to plates with initial stress by using the results obtained earlier for plates initially stress free.

Consider a multilayered plate constituted by n layers numbered from 1 to n. Each layer may be either homogeneous or composite with thin laminations, however in the latter case the average equivalent continuum of the particular layer considered is homogeneous. Consider the ith layer. The elastic coefficients M_i and L_i of this layer are constants as well as its couple-stress coefficient b_i. The average density ρ_i and the average initial stress P_i for this layer are also constant. Assuming perfect adherence we denote by τ_i and τ_{i+1} the shear stresses at the top and bottom of the ith layer. We recall that these stresses are assumed to be sinusoidally distributed in accordance with equations (7.4). It was shown[7, 9] that these stresses at three successive interfaces satisfy the following recurrence equations

$$B_i \tau_i + (A_i + A_{i+1})\tau_{i+1} + B_{i+1}\tau_{i+2} = -(c_i + c_{i+1})lV. \tag{8.0}$$

Using the approximation (7.16), i.e. $\mathfrak{M} = M$ the coefficients are defined as follows

$$A_i = \frac{a_i}{4\sqrt{M_i L_i}} \qquad B_i = \frac{a_i'}{4\sqrt{M_i L_i}}. \tag{8.1}$$

Furthermore

$$a_i = \tanh \beta_i \gamma_i + \frac{1}{\tanh \beta_i \gamma_i}$$

$$a'_i = \tanh \beta_i \gamma_i - \frac{1}{\tanh \beta_i \gamma_i} \tag{8.2}$$

$$c_i = \frac{1}{\beta_i} \tanh \beta_i \gamma_i$$

where

$$\beta_i = 2\sqrt{\frac{M_i}{L_i}} \qquad \gamma_i = \tfrac{1}{2} l h_i \tag{8.3}$$

$$h_i = \text{thickness of } i\text{th layer.}$$

We shall consider the case where the shear stresses τ_i and τ_{n+1} at the top and bottom faces of the plate are zero. The $n - 1$ equations (8.0) determine the $n - 1$ remaining stresses as linear functions of V. The static solution obtained earlier[7] is

$$q = \mathscr{S} = HV \tag{8.4}$$

where

$$H = -\sum^{i} (\tau_i + \tau_{i+1}) \frac{C_i}{V} + \sum^{i} h_i L_i l^2 \left(1 - \frac{c_i}{\gamma_i}\right). \tag{8.5}$$

Applying the basic analogy (7.13) the solution for the dynamic case with initial and couple stresses is

$$q = (H - \alpha^2 \rho_t - P_t l^2 + b_t l^4) V \tag{8.6}$$

where

$$\rho_t = \sum^{i} \rho_i h_i$$
$$P_t = \sum^{i} P_i h_i \tag{8.7}$$
$$b_t = \sum^{i} b_i h_i .$$

(*a*) *Free oscillations and buckling*

For $q = 0$ the plate is free of stresses at both faces, and (8.6) becomes the characteristic equation

$$H - \alpha^2 \rho_t - P_t l^2 + l_t l^4 = 0. \tag{8.8}$$

This equation yields the natural oscillation frequency α as a function of the initial stress P_t and the wave number l. If $p = i\alpha$ is real the deflection is proportional to the real exponential $\exp(pt)$ corresponding to dynamic buckling. For $\alpha = 0$ (8.8) yields the buckling load P_t as a function of the wavelength. Note that for a plate of finite span, which is simply supported at both ends, the span is equal to or a multiple of the half wavelength corresponding to either the fundamental mode or a higher harmonic.

Two- and three-layered plates. Equation (8.8) is immediately applicable to plates composed of two or three layers where one or more layers are constituted by a laminated composite. All that is needed is to substitute the value of H derived earlier[7] for the purely static problem without couple stresses.

Built-in end conditions. The foregoing solution assumes a sinusoidal solution along x. However in practice, a state of initial stress usually implies built-in end conditions. It was shown earlier that solutions satisfying such end conditions may be obtained readily using the foregoing results. This is accomplished by adding solutions which are exponential along the span. For example consider a sinusoidal solution of deflection

$$v = V \cos lx. \tag{8.9}$$

An exponential solution is obtained by replacing l by ik. We write

$$v = V_1 \cos ikx = V_1 \cosh kx. \tag{8.10}$$

Consider now the characteristic equation (8.8). We write it

$$H(l) - \alpha^2 \rho_t - P_t l^2 + b_t l^4 = 0 \tag{8.11}$$

where $H(l)$ denotes a function of l corresponding to the static sinusoidal solution. To the exponential solution corresponds a characteristic equation

$$H(ik) - \alpha^2 \rho_t + P_t k^2 + b_t k^4 = 0. \tag{8.12}$$

Elimination of α between equations (8.11) and (8.12) yields

$$H(l) - P_t l^2 + b_t l^4 = H(ik) + P_t k^2 + b_t k^4. \tag{8.13}$$

This determines k as a function of l. There are multiple branch solutions. Consider the branch of lowest value of k. An expression may then be written for v which is the sum of a sinusoidal and exponential solution

$$v(x) = V \cos lx + V_1 \cosh kx. \tag{8.14}$$

Corresponding values of u, are also obtained with arbitrary coefficients V and V_1. To end conditions may then be satisfied corresponding for example to $v = u = 0$ at both ends of the span. This yields a characteristic equation which provides the values of l, hence also of α by equation (8.11). The details of the procedure are the same as for the initially stress-free case[9] which was illustrated on examples. As already pointed out solutions of equation (8.13) for k may have an arbitrary number of branches, yielding a family of exponential solutions by which refined end conditions may be satisfied across the thickness of the plate.

In the case of static buckling $\alpha = 0$ the procedure is simplified. We eliminate P_t between equations (8.11) and (8.12) and obtain

$$\frac{1}{k^2} H(ik) + \frac{1}{l^2} H(l) + b_t(k^2 + l^2) = 0. \tag{8.15}$$

This determines k as a function of l. Again using a solution of the type (8.14) and introducing the built-in end conditions provides a characteristic equation for l. Equation (8.11) with $\alpha = 0$ then yields the buckling load P_t. Additional branch solutions of equation (8.15) for k provides ways of satisfying more refined end conditions across the thickness.

(*b*) *Forced oscillations*

The dynamic response for a loading of arbitrary distribution along the span and a plate simply supported at both ends is readily obtained by expanding the load in a Fourier series and applying equation (8.6) for each wavelength component.

However it may also be obtained by a more general method which is applicable to the case of a plate with built-in ends. Consider for example a fundamental mode of vibration for which l and k have been evaluated. The deflection of this mode is

$$v(x) = \mathrm{q} f(x) \tag{8.16}$$

where $f(x)$ is a normalized shape of the type (8.14) and q is an unknown amplitude. The mode shape $f(x)$ may be written

$$f(x) = \cos lx + R \cosh kx \tag{8.17}$$

where R is a fixed normalizing coefficient. The load required to maintain a static deflection $f(x)$ is derived from equation (7.13). It is written

$$\mathrm{q}\phi(x) = (\mathscr{H} \cos lx + R\mathscr{H}_1 \cosh kx)\mathrm{q} \tag{8.18}$$

where

$$\begin{aligned} \mathscr{H} &= H(l) - P_t l^2 + b_t l^4 \\ \mathscr{H}_1 &= H(ik) + P_t k^2 + b_t k^4. \end{aligned} \tag{8.19}$$

Note that equation (8.13) implies $\mathscr{H} = \mathscr{H}_1$. Obviously the elastic potential energy stored in the deformation is

$$\begin{aligned} \mathscr{V} &= \tfrac{1}{2}\mathrm{q}^2 Z \\ Z &= \int_{-s/2}^{s/2} f(x)\phi(x)\,\mathrm{d}s \end{aligned} \tag{8.20}$$

where the integral is evaluated over the span s. The kinetic energy is

$$\mathscr{T} = \tfrac{1}{2}T\dot{\mathrm{q}}^2 \tag{8.21}$$

where

$$T = \int_{-s/2}^{s/2} \rho_t f^2(x)\,\mathrm{d}x. \tag{8.22}$$

The Lagrangian equation for the normal mode considered is

$$T\ddot{\mathrm{q}} + Z\mathrm{q} = Q \tag{8.23}$$

where Q is the generalized force defined by the virtual work

$$Q\delta\mathrm{q} = \int_{-s/2}^{s/2} q(x, t) f(x)\delta\mathrm{q}\,\mathrm{d}x. \tag{8.24}$$

Hence

$$Q = \int_{-s/2}^{s/2} q(x, t) f(x)\,\mathrm{d}x \tag{8.25}$$

where $q(x, t)$ represents the distributed applied load as a function of time. Since the normal modes are uncoupled, the amplitudes of each mode are determined separately by this process.

(c) Damping of viscoelastic plates

In design analysis an important problem is the determination of the effect of viscoelastic layers on the vibration absorption at resonance. We assume the applied forces to be proportional to the harmonic timefactor $\exp(i\alpha t)$. In the case of the initially stress free plate a very simple procedure was developed[9] which may be readily carried over to the present case. When viscoelasticity is present the elastic coefficient M and L of each lamination are replaced by operators

$$\hat{M} = M + \Delta M$$
$$\hat{L} = L + \Delta L. \tag{8.26}$$

For harmonic oscillations they are separated into real parts M, L and imaginary parts ΔM ΔL. In the analysis advantage is taken of the fact that the real parts M and L vary only slowly with frequency and in a given range behave as constant elastic coefficients, while ΔM and ΔL are small. Each layer may be a composite made up of laminations, characterized by the operators

$$\hat{M}_1 = M_1 + \Delta M_1 \qquad \hat{M}_2 = M_2 + \Delta M_2$$
$$\hat{L}_1 = L_1 + \Delta L_1 \qquad \hat{L}_2 = L_2 + \Delta L_2 \tag{8.27}$$

where the imaginary parts are ΔM, ΔM_2, ΔL_1, ΔL_2. Applying equations (7.5) we find for the composite layer the following imaginary parts

$$\Delta M = \alpha_1 \Delta M_1 + \alpha_2 \Delta M_2$$
$$\Delta L = \left(\alpha_1 \frac{\Delta L_1}{L_1^{\,2}} + \alpha_2 \frac{\Delta L_2}{L_2^{\,2}}\right) L^2. \tag{8.28}$$

The quantities ΔL_1 ΔL_2 are assumed to be small. In this way the ith composite laminated layer is characterized by the operators

$$\hat{M}_i = M_i + \Delta M_i$$
$$\hat{L}_i = L_i + \Delta L_i \tag{8.29}$$

where ΔM_i and ΔL_i are expressed by equations (8.28). Similarly a couple-stress operator $\hat{b}_i$ for the ith laminated composite layer may be separated into real and imaginary parts

$$\hat{b}_i = b_i + \Delta b_i. \tag{8.30}$$

Consider first the case of a simply supported plate. By Fourier expansion the applied load may be considered as the superposition of sinusoidal distributions each associated with a certain value of l. For example for the fundamental mode this value corresponds to a wave-length equal to twice the span. The load $q \cos bx$ required to maintain a deflection $V \cos lx$ is given by equation (8.6), i.e.

$$q = (\hat{H} - \alpha^2 \rho_t - P_t l^2 + \hat{b}_t l^s)V. \tag{8.31}$$

At resonance we have the condition

$$H - \alpha^2 \rho_t - P_t l^2 + b_t l^4 = 0. \tag{8.32}$$

Hence q reduces to the purely imaginary quantity

$$\Delta q = (\Delta H + \Delta b_t l^4)V \tag{8.33}$$

where ΔH and Δb_t are the imaginary parts of $\hat{H}$ and $\hat{b}_t$. Note the value

$$\Delta b_t = \sum^{i} h_i \Delta b_i. \tag{8.34}$$

The force Δq is in phase with the velocity, and produces an amplitude V at resonance. It obviously constitutes a measure of the resonance damping.

It is of interest to point out that the *damping* as expressed by equation (8.33) is the *same as for the plate without initial stress.* Hence the results obtained previously[7, 9] for the particular cases of the two- and three-layered plates are immediately applicable to the plate with initial stress. In addition the term ΔH in equation (8.33) is readily derived from the purely static solution (8.4) without couple stresses.

For the plate with built-in end conditions a possible procedure is to use the normal modes as generalized coordinates and evaluate the damping of each mode separately. The method uses Lagrangian equations in operational form. It was applied to plates without initial stress[9]. In the present case we may proceed as follows. Replacing the elastic coefficients by complex quantities in expressions (8.18), the load distribution required to maintain the given modal amplitude (8.16) is

$$\mathrm{q}\hat{\phi}(x) = (\hat{\mathscr{H}} \cos lx + R\hat{\mathscr{H}}_1 \cosh kx)\mathrm{q} \tag{8.35}$$

where

$$\begin{aligned} \hat{\mathscr{H}} &= \mathscr{H} + \Delta\mathscr{H} \\ \hat{\mathscr{H}}_1 &= \mathscr{H}_1 + \Delta\mathscr{H}_1. \end{aligned} \tag{8.36}$$

The Lagrangian equation is then written

$$\hat{Z}\mathrm{q} - \alpha^2 T\mathrm{q} = Q \tag{8.37}$$

where

$$\hat{Z} = Z + \Delta Z = \int_{-s/2}^{s/2} \hat{\phi}(x) f(x)\, \mathrm{d}x \tag{8.38}$$

and

$$Q = \int_{-s/2}^{s/2} q(x) f(x)\, \mathrm{d}x \tag{8.39}$$

is the corresponding generalized force. In this expression $q(x)$ represents the applied harmonic load along the span without the time factor $\exp(i\alpha t)$. At resonance

$$Z - \alpha^2 T = 0. \tag{8.40}$$

Hence

$$\Delta Z\mathrm{q} = Q. \tag{8.41}$$

This equation yields the amplitude q at resonance under the applied generalized force Q. Using (8.35) and (8.38) we note that we may write

$$\Delta Z = \int_{-s/2}^{s/2} [\Delta\mathscr{H} \cos lx + R\Delta\mathscr{H}_1 \cosh kx] f(x)\, \mathrm{d}x. \tag{8.42}$$

This expression shows that $\mathrm{q}^2\, \Delta Z$ is the integrated product of the displacement $\mathrm{q}f(x)$ by the force required to maintain harmonic oscillations at resonance, hence it represents the power dissipated. Since $Q\mathrm{q}$ represents the power input, equation (8.41) may also be considered as expressing conservation of energy. It is of interest to write out the values of $\Delta\mathscr{H}$ and $\Delta\mathscr{H}_1$. From (8.19) we derive

$$\begin{aligned} \Delta\mathscr{H} &= \Delta H(l) + \Delta b_t l^4 \\ \Delta\mathscr{H}_1 &= \Delta H(ik) + \Delta b_t k^4. \end{aligned} \tag{8.43}$$

We note that expressions $\Delta H(l)$, $\Delta H(ik)$ are the same as those derived from the purely static solution without couple stresses. In this case we can see that the damping is affected only in a secondary way by the initial stress, namely through relation (8.13) between l and k.

9. THREE-DIMENSIONAL ANALYSIS

It was shown[9] how two-dimensional solutions provide readily corresponding solutions for three-dimensional problems for plates with transverse isotropic symmetry. The procedure may be extended to the present case provided the state of initial stress is also transverse isotropic. This implies

$$S_{11} = S_{33} \tag{9.1}$$

i.e. an isotropic initial stress in the xz plane. This stress may vary arbitrarily across the thickness. A constant initial stress component S_{22} may also be present normal to the plate.

For example the following trigonometric identity

$$v = \tfrac{1}{2}V[\cos(\xi x + \zeta z) + \cos(\xi x - \zeta z)] = V \cos \xi x \cos \zeta z \tag{9.2}$$

shows that a simply supported plate of rectangular shape in the xz plane behaves as the superposition of two plane strain solutions of wave number l given by

$$l^2 = \xi^2 + \zeta^2. \tag{9.3}$$

Solutions for plates of triangular plan forms are also derived by the same procedure[9].

Similarly a circular plate is analyzed by integrating plane strain solutions with equal weight for all directions around the y axis. We obtain for the deflection

$$v = V \int_0^{2\pi} \cos[lr \cos \theta] \, d\theta = 2\pi V J_0(lr) \tag{9.4}$$

where r is the radial coordinate and J_0 is Bessel's function. Another solution is obtained by changing l to ik,

$$v = 2\pi V_1 I_0(kr) \tag{9.5}$$

where I_0 is the modified Bessel's function. By superposition, a more general form is

$$v = 2\pi[V J_0(lr) + V_1 I_0(kr)]. \tag{9.6}$$

For natural oscillations k is a function of l, by equation (8.13). For buckling problems ($\alpha = 0$) this relation is replaced by equation (8.15). Boundary conditions provide a characteristic equation which determines l. This provides what we called an *intrinsic wavelength*[9]. The three-dimensional problem is then a superposition of plane strain solutions with this intrinsic wavelength. Evaluation of distribution of stress and displacement across the thickness for plane strain may be carried over readily to the three-dimensional case. The damping at resonance for the plane strain solution is *the same as in the three-dimensional problem of same intrinsic wavelength*. In this connection it is interesting to note the property derived in the foregoing approximate analysis that the damping is not affected significantly by the state of initial stress.

Attention is also called to the general nature of solutions of the type (9.6) which may contain more than one term of the type $I_0(kr)$ each with a different value of k as a branch function of l[9]. This provides the possibility of satisfying refined boundary conditions across the plate thickness.

Modes with non-circular symmetry may also be obtained by introducing weighting factors, functions of θ in the integral (9.4) or by taking successive partial derivatives $\partial/\partial x$ or $\partial/\partial y$ of the solution (9.6).

Problem of forced oscillations for three-dimensional problems with or without damping may be handled by using the normal modes as generalized coordinates as outlined in the preceding section and in earlier work[9].

10. EXACT ANALYSIS AND THE LIMITING CASE OF LARGE WAVELENGTH

We shall consider the case of a thinly laminated plate, such that the equivalent continuous material is homogeneous. The laminations are composed of alternating rigid and soft layers where average elastic coefficients are expressed by equations (4.4) while the couple stress coefficient b is given by equation (4.11). The plate is under an initial stress $P = -S_{11}$. We shall consider a bending deformation under a normal loading $q_1' \cos lx$ at the top face and $q_2' \cos lx = -q_1' \cos lx$ at the bottom face. We assume that tangential stresses vanish at both faces, hence $\tau_1 = \tau_2 = 0$. The normal displacements are equal at both faces, hence

$$V = V_1 = V_2 \tag{10.1}$$

while the tangential displacement is

$$U = U_1 = -U_2. \tag{10.2}$$

This is a special case of the general problem discussed in Section 6 and Appendix 2. Because of the couple-stress analogy the solution is formally identical to those obtained earlier[1][8] without couple stresses. The load is related to the displacements by the relations

$$\begin{aligned} 0 &= a_{11}U + a_{12}V \\ \frac{q'}{lL} &= a_{12}U + a_{22}V \end{aligned} \tag{10.3}$$

where

$$q' = q_1' = -q_2'. \tag{10.4}$$

Note that the load is applied antisymmetrically on top and bottom, so that the total load applied to the plate is

$$q = 2q'. \tag{10.5}$$

This total load is of course distributed sinusoidally along x. By eliminations of U in equations (10.3) we obtain

$$\frac{q}{2lL} = \frac{a_{11}a_{22} - a_{12}^2}{a_{11}} V \tag{10.6}$$

which relates directly the load to the deflection. The values of a_{ij} are given in Appendix 2. As already shown[8] (see also[1], p. 326) substitution of the values of a_{ij} leads to the result

$$\frac{a_{11}a_{22} - a_{12}^2}{a_{11}} = \frac{R_1 z_1 - R_2 z_2}{\Omega(\beta_1^2 - \beta_2^2)} \tag{10.7}$$

where

$$R_1 = \frac{(\Omega - C_{12}\beta_1^2)^2}{\Omega - L\beta_1^2}$$
$$R_2 = \frac{(\Omega + C_{12}\beta_2^2)^2}{\Omega - L\beta_2^2}. \tag{10.8}$$

This result is obtained after cancellation of the common factor (given by equation (2.8) of Appendix 2) in the numerator and denominator of expression (10.7). This result is valid for the dynamic case where the load q is multiplied by an harmonic function of time $\exp(i\alpha t)$. It also includes the presence of couple stresses.

It is of interest to examine the static case, without couple stresses, obtained by putting $\alpha = b = 0$ in equations (2.1) and (2.2) of Appendix 2. In particular we shall derive the limiting value of expression (10.7), in this case, for large wavelengths, i.e. for small values of l. This is obtained by expanding expression (10.7) in powers of l. For our purpose the expansion may be limited to the first two terms. This amounts to replacing z_1 and z_2 in expression (10.7) by the first two terms of their power expansion, i.e.

$$z_1 = \beta_1 \tanh \beta_1 \gamma = \beta_1(\beta_1 \gamma - \tfrac{1}{3}\beta_1^3\gamma^3)$$
$$z_2 = \beta_2 \tanh \beta_2 \gamma = \beta_2(\beta_2 \gamma - \tfrac{1}{3}\beta_2^3\gamma^3) \tag{10.9}$$

where $\gamma = \frac{1}{2}lh$ and h is the total plate thickness. We derive

$$\frac{a_{11}a_{22} - a_{12}^2}{a_{11}} = -\frac{P}{L}\gamma + \tfrac{1}{3}\gamma^3 \frac{C_{11}C_{22} - (C_{12} + P)^2}{LC_{22}}. \tag{10.10}$$

With this result the normal load q of equation (10.6) becomes

$$q = 2lV\left[-P\gamma + \tfrac{1}{3}\gamma^3 \frac{C_{11}C_{22} - (C_{12} + P)^2}{C_{22}}\right]. \tag{10.11}$$

Buckling instability is obtained by putting $q = 0$. This yields

$$P = \tfrac{1}{3}\gamma^2 \left[\frac{C_{11}C_{22} - (C_{12} + P)^2}{C_{22}}\right]. \tag{10.12}$$

Since P is of the order γ^2 we may write the limiting value for γ small as

$$P = \tfrac{1}{3}\gamma^2 \frac{C_{11}C_{22} - C_{12}^2}{C_{22}}. \tag{10.13}$$

With the definition (7.3) of M this becomes

$$P = \frac{4M}{3}\gamma^2. \tag{10.14}$$

This coincides with the critical buckling load derived from the Euler theory for a thin plate under a compressive stress P, of span equal to half the wavelength and simply supported at both ends (see[1] p. 231).

Note that the foregoing results are applicable for a material with couple-stresses, since the couple-stress analogy remains valid for the limiting case. Hence for this case all we need to do is to replace P by

$$P - bl^2 = P - \frac{4b}{h^2}\gamma^2 \tag{10.15}$$

in equations (10.11)–(10.14). For example (10.14) becomes

$$P = 4\left(\tfrac{1}{3}M + \frac{b}{h^2}\right)\gamma^2 \tag{10.16}$$

of course in practice, the term b/h^2 constitutes only a small correction, as is to be expected for large wavelengths.

11. NUMERICAL COMPARISON OF EXACT AND APPROXIMATE BUCKLING ANALYSIS

As in the preceding section we shall consider the case of a thinly laminated plate, such that the equivalent continuous material is homogeneous. We shall evaluate the statical buckling load P (hence putting $\alpha = 0$), as a function of the wavelength. In order to simplify the presentation we first assume the couple stress to be negligible ($b = 0$).

According to the approximate analysis of Section 8 we apply equation (8.8) putting $\alpha = b_t = 0$. This yields

$$P_t l^2 = Phl^2 = H \tag{11.1}$$

where $P = P_t/h$ is the compressive load per unit area. Since we are dealing with a single homogeneous layer of thickness h with free faces ($\tau_1 = \tau_2 = 0$), the value of (8.5) of H becomes

$$H = Lhl^2\left(1 - \frac{\tanh \beta\gamma}{\beta\gamma}\right) \tag{11.2}$$

where $\beta = 2\sqrt{M/L}$, $\gamma = \tfrac{1}{2}lh$. With this value equation (11.1) yields

$$\frac{P}{L} = 1 - \frac{\tanh \beta\gamma}{\beta\gamma} \tag{11.3}$$

where P is the critical buckling load. The buckling wavelength is $\mathscr{L} = \pi h/\gamma$. Note that

$$\beta\gamma = 2\sqrt{\frac{M}{L}}\,\gamma. \tag{11.4}$$

If

$$2\sqrt{\frac{M}{L}}\,\gamma \ll 1$$

we write approximately

$$\tanh \beta\gamma = \beta\gamma - \tfrac{1}{3}\beta^3\gamma^3 \tag{11.5}$$

and equation (11.3) becomes

$$P = \tfrac{4}{3}M\gamma^2. \tag{11.6}$$

This coincides with the value (10.14) derived from the Euler theory for thin plates.

Note that the approximate value of the buckling load given by (11.3) is valid for either compressible or incompressible materials. In order to evaluate the accuracy of this approximate value (11.3) we shall compare it with exact results for the case of an incompressible material. The reason for this choice is that the numeral evaluation of the exact value is much simpler for this case. Furthermore if the accuracy is valid for the incompressible case it should be also valid for the more general compressible case.

The exact theory for the incompressible case has been developed and discussed extensively by the author[1, 3, 8]. For plane strain the two-dimensional stress–strain relation of an incompressible material are written[1, 3, 8]

$$\begin{aligned} s_{11} - s &= 2Ne_{xx} \\ s_{22} - s &= 2Ne_{yy} \\ s_{12} &= 2Qe_{xy} \end{aligned} \tag{11.7}$$

where $s = \frac{1}{2}(s_{11} + s_{22})$, while s_{ij} are the incremental stresses referred to final areas after deformation. In terms of the stresses t_{11} t_{22} t'_{12} using relations (3.4) we derive

$$\begin{aligned} t_{11} - t_{22} &= 4Me_{xx} \\ t'_{12} &= 2Le_{xy} \end{aligned} \tag{11.8}$$

with the condition

$$e_{xx} + e_{yy} = 0 \tag{11.9}$$

and putting

$$M = N + \tfrac{1}{4}P \qquad L = Q + \tfrac{1}{2}P. \tag{11.10}$$

The value of $(a_{11}a_{22} - a'_{12})/a_{11}$ for the incompressible case as derived in previous work[1, 3, 8] is

$$\frac{a_{11}a_{22} - a_{12}^2}{a_{11}} = \frac{(\beta_1^2 + 1)^2 z_2 - (\beta_2^2 + 1)^2 z_1}{\beta_1^2 - \beta_2^2} \tag{11.11}$$

with

$$\begin{aligned} \beta_1^2 &= m + \sqrt{m^2 - k^2} \\ \beta_2^2 &= m - \sqrt{m^2 - k^2} \\ m &= \frac{2M}{L} - 1 \qquad k^2 = 1 - \frac{P}{L}. \end{aligned} \tag{11.12}$$

The values of z_1 and z_2 are expressed by (2.4) of Appendix 2. In earlier work[1, 8] we have also shown how these results may be derived by a limiting process starting from the more general elastic coefficients (3.8) for a compressible material. It is readily verified that the coefficient M in (11.8) is the limiting value of (7.3) for the case of incompressibility.

The exact characteristic equation for buckling is obtained by substituting the value (11.11) in equation (10.6) and putting $q = 0$. We obtain

$$(\beta_1^2 + 1)^2 z_2 - (\beta_2^2 + 1)^2 z_1 = 0. \tag{11.13}$$

The unknown in this equation is P/L which must be determined in terms of the two parameters

$$\frac{M}{L} = \tfrac{1}{4}\beta^2 \qquad \gamma = \tfrac{1}{2}lh. \tag{11.14}$$

The first may be considered as a measure of the anisotropy, while the second may be written

$$\gamma = \frac{\pi h}{\mathscr{L}} \tag{11.15}$$

where $\mathscr{L}$ is the buckling wavelength. For example $\gamma = 1$, when $\mathscr{L} = \pi h$, hence for a wavelength about three times the plate thickness. Numerical values are shown in Table 1.

Table 1

$\beta\gamma$	P/L (a)	P/L (b) $\beta = 2$	P/L (c) $\beta = 4$	P/L (d) $\beta = 6$
1	0·2384	0·2425	0·2395	0·2385
4	0·7501	0·7755	0·7575	0·7535
10	0·9000	0·9055	0·9044	0·9015

Column (a), Approximate values by equation (11.3); Column (b), Exact values by equation (11.13) for $\beta = 2$; Column (c), Exact values for $\beta = 4$; Column (d), Exact values for $\beta = 6$.

As predicted by the approximate theory (11.3) it is verified that P/L depends mainly on the product $\beta\gamma$. Comparison of the approximate values of column (a) with exact values of columns (b–d) for $\beta = 2, 4, 6$ shows excellent agreement. We remember that the approximate value (11.3) is valid for compressible materials, for which the exact evaluation leads to very tedious numerical work. The very simple approximate theory provides therefore an enormous simplification especially when applied to physically more sophisticated cases of multilayered plates with different types of homogeneous or laminated layers and complicated boundary conditions.

It is interesting to introduce at this point another related approximation. We may write the approximation

$$1 - \frac{\tanh \beta\gamma}{\beta\gamma} = \frac{\beta^2\gamma^2}{3 + \kappa\beta^2\gamma^2} \tag{11.16}$$

where κ is almost constant. It varies slowly as function of $\beta\gamma$ according to Table 2.

Table 2

$\beta\gamma$	1	2	4	10	30	∞
κ	1·18	1·18	1·14	1·08	1·03	1·00

The approximation (11.16) was proposed earlier[7] using a constant value $\kappa = 1{\cdot}18$ in which case it remains valid for practical purpose in the range $\beta\gamma < 5$.

With the approximation (11.16) relation (11.3) may be written

$$Pl^2 + \tfrac{1}{3}\kappa \frac{M}{L} h^2 l^4 = \tfrac{1}{3} M h^2 l^4. \tag{11.17}$$

For a deflection v proportional to $\cos lx$ this equation is equivalent to the differential equation

$$-P\frac{\partial^2 v}{\partial x^2}+\tfrac{1}{3}\kappa\frac{M}{L}h^2\frac{\partial^4 v}{\partial x^4}=\tfrac{1}{3}Mh^2\frac{\partial^4 v}{\partial x^4}. \tag{11.18}$$

This is the same as obtained from the "Timoshenko beam" theory where the term containing κ is due to the transversal shearing deformation. In the Timoshenko theory the value of κ is treated as a constant to be determined from elasticity theory. Its correct value given by Table 2 shows that it is almost constant of value between 1·18 and 1·00 while in the range $\beta\gamma < 5$ it may be put equal to the constant value 1·18.

The case where couple-stresses become significant is readily obtained by using the couple stress analogy introduced earlier[4] and discussed in more detail in Section 6. According to this analogy we simply replace P by $P - bl^2$ where b is the couple-stress coefficient. Equation (11.3) for the critical buckling load becomes

$$P=\left(1-\frac{\tanh\beta\gamma}{\beta\gamma}\right)L+\frac{4b}{h^2}\gamma. \tag{11.19}$$

The significant features of this result were already discussed qualitatively earlier[4]. With the approximation (11.16) the buckling load (11.19) becomes

$$P=\frac{1}{3}\frac{Ml^2h^2}{1+\frac{1}{3}\kappa(M/L)l^2h^2}+bl^2. \tag{11.20}$$

With $\kappa = 1$ this is essentially the same result as obtained in an earlier discussion of a folding problem of multilayers in geology[10]. The transition from bending buckling to shear buckling appears in the *transition wavelength* range [1, 7, 10] determined approximately by the equation

$$\frac{1}{3}\frac{M}{L}l^2h^2=1. \tag{11.21}$$

For wavelengths larger than the value thus determined, the buckling involves mainly a bending mechanism.

It should be noted that the range of applicability of the approximate theory of Sections 7 and 8 extends considerably beyond the "Timoshenko beam" type of approximation since the latter overlooks the "skin effect"[7] which plays an important role at interfaces of multilayered plates and may have a considerable influence on the damping for viscoelastic materials.

The present relatively simple problem provides a good illustration of the "basic analogy" discussed in Section 7 as a direct consequence of equation (7.13). Under a load q the static solution (8.4) is

$$q=\mathscr{S}=HV. \tag{11.22}$$

With the value (11.2) this becomes

$$q=Lhl^2\left(1-\frac{\tanh\beta\gamma}{\beta\gamma}\right)V \tag{11.23}$$

a result already derived earlier[7]. According to the basic analogy we replace q by $q + \mathscr{R}V$, where $\mathscr{R}$ is expressed by (7.14). Hence (11.23) becomes

$$\frac{q}{hV} = Ll^2\left(1 - \frac{\tanh \beta\gamma}{\beta\gamma}\right) - Pl^2 - \alpha^2\rho + bl^4. \tag{11.24}$$

This expresses the normal load on the plate with couple-stresses, initial stress P and forced oscillations proportional to the harmonic factor $\exp(i\alpha t)$. The characteristic equation for dynamic buckling is obtained by putting $q = 0$ and $i\alpha = p$ real and positive. The same procedure applies to viscoelastic creep bucklıng where M and L are functions of p instead of $i\alpha$.

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APPENDIX 1

Various alternate derivations of the equilibrium equations

There are several procedures which may be used to derive the basic equilibrium equations of the incremental stress field.

1. *Locally rotated axes.* One procedure is to refer the stresses to locally rotated axes. As already pointed out[1, 5] the rotation is not determined uniquely and a certain amount of arbitrariness is left in defining this rotation in the manner most suitable to the physics of the problem as will be illustrated below. Consider a state of initial stress $S_{11}(y)S_{22}$ and S_{12}. For the sake of completeness we assume here the presence of an initial shear stress S_{12}. Both S_{22} and S_{12} are constant while S_{11} depends on y. In the deformation the initial coordinates x, y become $\xi = x + u$, $\eta = y + v$. Hence plane strain is assumed. A fourth

component S_{33} of initial stress normal to the plane of deformation may of course be present but does not appear explicitly in the formulation. We denote by $\bar{\sigma}_{\xi\xi}\ \bar{\sigma}_{\eta\eta}\ \bar{\sigma}_{\xi\eta}$ the stress components per unit final area at the point ξ, η and referred to fixed axes x, y. The equilibrium equations of this field are

$$\begin{aligned} \frac{\partial \bar{\sigma}_{\xi\xi}}{\partial \xi} + \frac{\partial \bar{\sigma}_{\xi\eta}}{\partial \eta} &= \rho' \frac{\partial^2 u}{\partial t^2} \\ \frac{\partial \bar{\sigma}_{\xi\eta}}{\partial \xi} + \frac{\partial \bar{\sigma}_{\eta\eta}}{\partial \eta} &= \rho' \frac{\partial^2 v}{\partial t^2} \end{aligned} \tag{1.1}$$

where ρ' is the mass per unit volume after deformation. If ρ denotes the mass per unit initial volume we may write

$$\rho = J\rho' \tag{1.2}$$

where J is the Jacobian

$$J = \begin{vmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \xi}{\partial y} \\ \dfrac{\partial \eta}{\partial x} & \dfrac{\partial \eta}{\partial y} \end{vmatrix}. \tag{1.3}$$

We may also express the derivatives $\partial/\partial\xi$, $\partial/\partial\eta$ in terms of the derivatives $\partial/\partial x$, $\partial/\partial y$ with respect to the original coordinates. The transformation equations are

$$\begin{aligned} \frac{\partial}{\partial \xi} &= \frac{1}{J}\left(\frac{\partial \eta}{\partial y}\frac{\partial}{\partial x} - \frac{\partial \eta}{\partial x}\frac{\partial}{\partial y}\right) \\ \frac{\partial}{\partial \eta} &= \frac{1}{J}\left(-\frac{\partial \xi}{\partial y}\frac{\partial}{\partial x} + \frac{\partial \xi}{\partial x}\frac{\partial}{\partial y}\right). \end{aligned} \tag{1.4}$$

We now refer the stress to locally rotated axes. At each displaced point we rotate the reference axes of the stress through a small angle θ. The angle θ is chosen in such a way that in the absence of local deformation of the material the stress components referred to the rotated axes remain constant and equal to the initial stress. If a deformation is present the stress components referred to the rotated axes become

$$\begin{aligned} \sigma_{11} &= S_{11} + \Delta\sigma_{11} \\ \sigma_{22} &= S_{22} + \Delta\sigma_{22} \\ \sigma_{12} &= S_{12} + \Delta\sigma_{12}. \end{aligned} \tag{1.5}$$

The incremental stresses $\Delta\sigma_{11}\ \Delta\sigma_{22}\ \Delta\sigma_{12}$ are due only to the strain. We shall assume that they are small of the first order. In that case neglecting higher order quantities we may write

$$\begin{aligned} \bar{\sigma}_{\xi\xi} &= S_{11} + \Delta\sigma_{11} - 2S_{12}\theta \\ \bar{\sigma}_{\eta\eta} &= S_{22} + \Delta\sigma_{22} + 2S_{12}\theta \\ \bar{\sigma}_{\xi\eta} &= S_{12} + \Delta\sigma_{12} - P\theta \end{aligned} \tag{1.6}$$

where

$$P = S_{22} - S_{11}(y). \tag{1.7}$$

We now substitute the values (1.6) and the differential operators (1.4) into equations (1.1) neglecting higher order terms. This yields

$$\frac{\partial}{\partial x}\Delta\sigma_{11} + \frac{\partial}{\partial y}\Delta\sigma_{12} - 2S_{12}\frac{\partial\theta}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial P}{\partial y} - \frac{\partial}{\partial y}(P\theta) = \rho\frac{\partial^2 u}{\partial t^2}$$
$$\frac{\partial}{\partial x}\Delta\sigma_{12} + \frac{\partial}{\partial y}\Delta\sigma_{22} + 2S_{12}\frac{\partial\theta}{\partial y} - P\frac{\partial\theta}{\partial x} = \rho\frac{\partial^2 v}{\partial t^2}. \tag{1.8}$$

As already stated the choice of the rotation θ is not unique. One possible choice is to define θ as

$$\theta = \frac{\partial v}{\partial x}. \tag{1.9}$$

This angle represents the material rotation of a linear element originally coincident with the x direction. If the material is laminated along this direction θ represents the local rotation of the thin layers. We denote by $s'_{11} = \Delta\sigma_{11}$ $s'_{22} = \Delta\sigma_{22}$ and $s'_{12} = \Delta\sigma_{12}$ the incremental stresses for this particular choice of rotation. The equilibrium equations (1.8) become

$$\frac{\partial s'_{11}}{\partial x} + \frac{\partial s'_{12}}{\partial y} - 2S_{12}\frac{\partial^2 v}{\partial x^2} - P\frac{\partial^2 v}{\partial x\,\partial y} = \rho\frac{\partial^2 u}{\partial t^2}$$
$$\frac{\partial s'_{12}}{\partial x} + \frac{\partial s'_{22}}{\partial y} + 2S_{12}\frac{\partial^2 v}{\partial x\,\partial y} - P\frac{\partial^2 v}{\partial x^2} = \rho\frac{\partial^2 v}{\partial t^2}. \tag{1.10}$$

Consider the particular case $S_{12} = 0$. We put (see equation 3.1)

$$t_{11} = s'_{11} - Pe_{yy}$$
$$t_{22} = s'_{22} \tag{1.11}$$
$$t'_{12} = s'_{12}.$$

Equations (1.10) are now written

$$\frac{\partial t_{11}}{\partial x} + \frac{\partial t'_{12}}{\partial y} = \rho\frac{\partial^2 u}{\partial t^2}$$
$$\frac{\partial t'_{12}}{\partial x} + \frac{\partial t_{22}}{\partial y} = \rho\frac{\partial^2 v}{\partial t^2} + P\frac{\partial^2 v}{\partial x^2}. \tag{1.12}$$

They coincide with equations (2.5) of Section 2 and the variables have the same physical significance.

As shown in previous work[1, 2] the value

$$\theta = \omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{1.13}$$

constitutes another possible choice for the local rotation. The incremental stresses in this case were denoted by $\Delta\sigma_{11} = s_{11}$, $\Delta\sigma_{22} = s_{22}$, $\Delta\sigma_{12} = s_{12}$. For the particular case $S_{12} = 0$ the equilibrium equations (1.8) become

$$\frac{\partial s_{11}}{\partial x} + \frac{\partial s_{22}}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial P}{\partial y} - \frac{\partial}{\partial y}(P\omega) = \rho\frac{\partial^2 u}{\partial t^2}$$
$$\frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} - P\frac{\partial\omega}{\partial x} = \rho\frac{\partial^2 v}{\partial t^2}. \tag{1.14}$$

Proceeding as before[1, 4] (see equation 3.4) we substitute the values

$$\begin{aligned} s_{11} &= t_{11} + Pe_{yy} \\ s_{22} &= t_{22} \\ s_{12} &= t'_{12} - Pe_{xy}. \end{aligned} \tag{1.15}$$

With these values we derive again equations (2.5) of Section 2 with the same physical interpretation of the stresses.

2. *Method of virtual work.* In the context of the method of virtual work the definition of cartesian strain relative to rotated axes may be chosen in various ways. This was pointed out in the authors book[1] and further developed in greater detail in a recent paper[5]. For example in the present case the strain components are defined relative to axes, which are rotated through the angle $\theta = \partial v/\partial x$ which coincides with the angle of rotation of the laminations. In this formulation we first apply the local differential transformation

$$\begin{aligned} d\xi &= (1 + \varepsilon_{11})\, dx + 2\varepsilon_{12}\, dy \\ d\eta &= (1 + \varepsilon_{22})\, dy. \end{aligned} \tag{1.16}$$

Hence the deformation is such that the material points on the x axis remain on this axis. We then rotate the material without additional deformation through the angle $\theta = \partial v/\partial x$. After this rotation the transformation becomes the actual differential local transformation of the material, i.e.

$$\begin{aligned} d\xi &= (1 + a_{11})\, dx + a_{12}\, dy \\ d\eta &= a_{21}\, dx + (1 + a_{22})\, dy \end{aligned} \tag{1.17}$$

where

$$\begin{aligned} a_{11} &= \frac{\partial u}{\partial x} \qquad a_{12} = \frac{\partial u}{\partial y} \\ a_{21} &= \frac{\partial v}{\partial x} \qquad a_{22} = \frac{\partial v}{\partial y}. \end{aligned} \tag{1.18}$$

We then evaluate

$$ds^2 = d\xi^2 + d\eta^2 \tag{1.19}$$

first by substituting the values (1.16) and then the values (1.17). Since they both represent the same deformation the coefficients of dx^2, $dx\, dy$ and dy^2 must be the same for the two results. This yields the three equations

$$\begin{aligned} \varepsilon_{11} + \tfrac{1}{2}\varepsilon_{11}^2 &= a_{11} + \tfrac{1}{2}(a_{11}^2 + a_{21}^2) \\ \varepsilon_{12} + \varepsilon_{12}\varepsilon_{11} &= \tfrac{1}{2}(a_{12} + a_{21}) + \tfrac{1}{2}(a_{11}a_{12} + a_{22}a_{21}) \\ \varepsilon_{22} + \tfrac{1}{2}\varepsilon_{22}^2 + 2\varepsilon_{12}^2 &= a_{22} + \tfrac{1}{2}(a_{22}^2 + a_{12}^2). \end{aligned} \tag{1.20}$$

If a_{ij} is of the first order, a first order solution is $\varepsilon_{11} = a_{11}$ $\varepsilon_{22} = a_{22}$ $\varepsilon_{12} = \frac{1}{2}(a_{12} + a_{21})$. A second order solution is obtained by substituting the first order values in the quadratic terms on the left side of (1.20). We derive

$$\begin{aligned} \varepsilon_{11} &= a_{11} + \tfrac{1}{2}a_{21}^2 \\ \varepsilon_{12} &= \tfrac{1}{2}(a_{12} + a_{21}) + \tfrac{1}{2}a_{21}(a_{22} - a_{11}) \\ \varepsilon_{22} &= a_{22} - \tfrac{1}{2}a_{21}(2a_{12} + a_{21}). \end{aligned} \tag{1.21}$$

We denote by

$$\tau'_{11} = S_{11} + t'_{11} \qquad \tau'_{22} = S_{22} + t'_{22} \qquad \tau'_{12} = S_{12} + t'_{12} \tag{1.22}$$

the stresses, i.e. the forces per unit initial area along directions parallel to the rotated axes. The terms t'_{11} t'_{22} and t'_{12} are the incremental stresses, while τ'_{12} is the force acting on a face initially parallel to the x direction. The principal of virtual work is written[1]

$$\int (\tau'_{11}\,\delta\varepsilon_{11} + \tau'_{22}\,\delta\varepsilon_{22} + 2\tau'_{12}\,\delta\varepsilon_{12} - \rho\ddot{u}\,\delta u - \rho\ddot{v}\,\delta v)\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{1.23}$$

This being valid for arbitrary variations δu δv yields the following differential equations

$$\begin{aligned} \frac{\partial t'_{11}}{\partial x} + \frac{\partial t'_{12}}{\partial y} - S_{12}\frac{\partial^2 v}{\partial x^2} - S_{22}\frac{\partial^2 v}{\partial x\,\partial y} &= \rho\ddot{u} \\ \frac{\partial t'_{12}}{\partial x} + \frac{\partial t'_{22}}{\partial y} - 2S_{22}\frac{\partial e_{xy}}{\partial x} + S_{12}\frac{\partial}{\partial x}(e_{yy} - e_{xx}) + S_{12}\frac{\partial^2 v}{\partial x\,\partial y} + S_{11}\frac{\partial^2 v}{\partial x^2} &= \rho\ddot{v}. \end{aligned} \tag{1.24}$$

It is easily shown that these equations are equivalent to the previously derived results. The following relations between the stresses are derived either analytically or by physical reasoning

$$\begin{aligned} t'_{11} &= s'_{11} + S_{11}e_{yy} - 2S_{12}e_{xy} \\ t'_{22} &= s'_{22} + S_{22}e_{xx} \\ t'_{12} &= s'_{12} + S_{12}e_{xx}. \end{aligned} \tag{1.25}$$

If we substitute these values into equations (1.24) we obtain the previous equations (1.10). Note that the physical significance of the equilibrium equations (1.24) are brought out by putting

$$t'_{21} = t'_{12} - 2S_{22}e_{xy} + S_{12}(e_{yy} - e_{xx}). \tag{1.26}$$

This quantity is the incremental stress acting on the face initially normal to x, and in a direction normal to x. Equation (1.26) expresses the condition of equilibrium of moments on a unit element.

With these variables equations (1.24) become

$$\begin{aligned} \frac{\partial t'_{11}}{\partial x} + \frac{\partial t'_{12}}{\partial y} - S_{12}\frac{\partial^2 v}{\partial x^2} - S_{22}\frac{\partial^2 v}{\partial x\,\partial y} &= \rho\ddot{u} \\ \frac{\partial t'_{21}}{\partial x} + \frac{\partial t'_{22}}{\partial y} + S_{12}\frac{\partial^2 v}{\partial x\,\partial y} + S_{11}\frac{\partial^2 v}{\partial x^2} &= \rho\ddot{v}. \end{aligned} \tag{1.27}$$

The terms in these equations yield an obvious physical interpretation.

APPENDIX 2

Evaluation of the six matrix elements for a plate with couple stresses

The six coefficients A B C D E F were evaluated earlier[1, 8], in the absence of couple stresses ($b = 0$) and in terms of the elastic coefficients B_{11} B_{22} B_{21}. These values are immediately extended to the present case. According to equations (3.8) we replace B_{11} B_{22}

and B_{12} by C_{11} C_{22} and C_{12}. Furthermore as a consequence of the couple-stress analogy we replace P by $P - bl^2$. The following results are obtained. We put

$$2m = \frac{1}{LC_{22}}\left[\Omega C_{22} - L\left(2C_{12} + P - bl^2 + \frac{\alpha^2\rho}{l^2}\right) - C_{12}^2\right]$$
$$k^2 = \frac{\Omega}{LC_{22}}\left(L - P + bl^2 - \frac{\alpha^2\rho}{l^2}\right) \tag{2.1}$$

with

$$\Omega = C_{11} - \frac{\alpha^2\rho}{l^2}. \tag{2.2}$$

Next we put

$$\beta_1^2 = m + \sqrt{m^2 - k^2}$$
$$\beta_2^2 = m - \sqrt{m^2 - k^2} \tag{2.3}$$

choosing values of β_1 and β_2 such that the real part is nonnegative. Also we write

$$z_1 = \beta_1 \tanh \beta_1\gamma \qquad z_2 = \beta_2 \tanh \beta_2\gamma$$
$$z_1' = \frac{1}{\beta_1}\tanh \beta_1\gamma \qquad z_2' = \frac{1}{\beta_2}\tanh \beta_2\gamma \tag{2.4}$$

where

$$\gamma = \tfrac{1}{2}lh \qquad (h = \text{plate thickness}) \tag{2.5}$$

is a non-dimensional wave-number.

With these definitions we introduce the following six dimensionless elements

$$a_{11} = \Omega(\beta_2^2 - \beta_1^2)\frac{1}{\Lambda_a}$$
$$a_{22} = C_{22}(\beta_2^2 - \beta_1^2)z_1 z_2\frac{1}{\Lambda_a} \tag{2.6}$$
$$a_{12} = [(\Omega + C_{12}\beta_2^2)z_1 - (\Omega + C_{12}\beta_1^2)z_2]\frac{1}{\Lambda_a}$$

$$b_{11} = \Omega(\beta_2^2 - \beta_1^2)z_1' z_2'\frac{1}{\Lambda_s}$$
$$b_{22} = C_{22}(\beta_2^2 - \beta_1^2)\frac{1}{\Lambda_s} \tag{2.7}$$
$$b_{12} = [(\Omega + C_{12}\beta_2^2)z_2' - (\Omega + C_{12}\beta_1^2)z_1']\frac{1}{\Lambda_s}$$

where

$$\Lambda_a = (\Omega - L\beta_1^2)z_2 - (\Omega - L\beta_2^2)z_1$$
$$\Lambda_s = (\Omega - L\beta_1^2)z_1' - (\Omega - L\beta_2^2)z_2'. \tag{2.8}$$

Finally we obtain the six matrix elements of the plate as

$$\begin{aligned} A &= \tfrac{1}{2}(a_{11} + b_{11}) \qquad & D &= \tfrac{1}{2}(a_{11} - b_{11}) \\ B &= \tfrac{1}{2}(a_{12} + b_{12}) \qquad & E &= \tfrac{1}{2}(a_{12} - b_{12}) \\ C &= \tfrac{1}{2}(a_{22} + b_{22}) \qquad & F &= \tfrac{1}{2}(a_{22} - b_{22}). \end{aligned} \tag{2.9}$$

Резюме — Разрабатывается механика сплошных сред многослойных листов под начальным напряжением и включается случай, где некоторые или все слои состоят из очень тонко расслоенных материалов, находящихся под влиянием пары деформирующих сил. Это применимо к вопросам коробления, динамики и вибраций, и включает эволюцию упруговязкой ползучести, продольный изгиб и абсорбцию вибраций. Получили два вида результатов. Один результат получили благодаря строгому анализу и общей теории разностной деформации, а другой — введением радикальных упрощений, но в то же время, сохранением существенных физических свойств, являющимися поведением сплошных сред. Подчеркивается применение определенного дифференциального напряжения, что при обсуждаемом вопросе является значительным преимуществом. Формулируются также соответствующие вариационные принципы и сравниваются как аналитически, так и численно точные и приблизительные теории. Сравнения успешно совпали.